[7]Adamczyk, J. J., "Model Equation for Simulating Flows in Multistage Turbomachinery," NASA TM-86869, Nov. 1984.

[8]Mulac, R. A., and Adamczyk, J. J., "The Numerical Simulation of a High-Speed Axial Flow Compressor," *ASME Journal of Turbomachinery,* Vol. 114, No. 3, 1992, pp. 517-527.

[9]Denton, J. D., "The Calculation of Three-Dimensional Viscous Flow Through Multistage Turbomachines," *ASME Journal of Turbomachinery,* Vol. 114, No. 1, 1992, pp. 18-26.

[10]Dawes, W. N., "Toward Improved Throughflow Capability: The Use of Three-Dimensional Viscous Flow Solvers in a Multistage Environment," *ASME Journal of Turbomachinery,* Vol. 114, No. 1, 1992, pp. 8-17.

[11]Zhang, X., "Passage-Averaged Approximation of Turbomachinery Flow Using Vorticity-Potential Method," Ph.D. Thesis, Dept. of Applied Mathematics, École Polytechnique de Montréal, Canada, July 1991.

[12]Hirasaki, G. J., and Hellums, J. D., "Boundary Conditions on the Vector and Scalar Potentials in Viscous Three-Dimensional Hydrodynamics," *Quarterly of Applied Mathematics,* Vol. 28, No. 2, 1970, pp. 293-296.

[13]Carey, C., Fraser, S. M., Rachman, D., and Wilson, G., "Studies of the Flow of Air in a Model Mixed-Flow Pump Using Laser Doppler Anemometry," Technical Rept., National Engineering Lab. Rept. No. 698-699, Glasgow, Scotland, July 1985.

Asymptotic Probability Density Function of a Scalar

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Introduction

THERE are inherent difficulties in measuring contaminant statistics in general (see Derksen and Sullivan[1]) and conditional probabilities in specific (see Chatwin and Sullivan[2]). In Chatwin and Sullivan[3] an expression for the moments of the probability density function (PDF), given in Eq. (1), was shown to represent experimental data from a wide variety of turbulent shear flows. The data were taken at large enough downstream distances in these flows that the moments of the scalar concentration appeared to be self-similar. The objective here is to find the self-similar PDF of contaminant concentration that corresponds to those moments.

The asymptotic moments of the PDF are given in Chatwin and Sullivan[3] as

$$\mu_n' = \frac{\beta^n}{\alpha}\mu_1^n(0)[\chi(\alpha-\chi)^n + (-1)^n(\alpha-\chi)\chi^n] \tag{1}$$

where $\chi = \mu_1(\eta)/\mu_1(0)$, $\eta = y/\ell = 0$ is evaluated at the location of the maximum mean concentration (the centerline of flows like jets and wakes and at the wall in boundary layers), and y is the cross-stream distance measured from this position with ℓ defined by $\mu_1(\ell) = \frac{1}{2}\mu_1(0)$. Here,

$$\mu_n = \int_0^\infty \theta^n p(\theta)\,\mathrm{d}\theta, \qquad \mu_n' = \int_0^\infty (\theta-\mu_1)^n p(\theta)\,\mathrm{d}\theta \tag{2}$$

and

$$p(\theta) = \mathrm{prob}(\theta \le \gamma < \theta + \mathrm{d}\theta) \tag{3}$$

is the PDF for scalar contaminant concentration θ. Equation (1) has the advantage that all of the central moments are expressed in terms of the mean concentration μ_1, which is the most easily measured and theoretically predicted statistic. That is, μ_1 is relatively insensitive to temporal and spatial measurement averaging and to the effects of molecular diffusivity. The distribution of lower ordered moments, at normally measured downstream distances, appears to be reasonably described by Eq. (1) over a wide range of flow conditions including jets, wakes, boundary layers, and plumes in grid turbulence. The range of the two parameters α and β that appear in Eq. (1) was found to be relatively narrow, i.e., $1 \le \alpha \le 1.5$ and $0.1 \le \beta \le 0.75$. In Derksen and Sullivan[1] the practical advantages of using measured lower ordered moments is discussed. There a Jacobi polynomial expansion, motivated by the structure of the lower ordered moments in Eq. (1), and also a maximum entropy formulation were used to invert measured moments to generate the PDF. The asymptotic moments given by Eq. (1) are not amenable to inversion using Jacobi polynomials; however, careful application of the maximum entropy formulation enables one to compile the asymptotic forms of the PDF.

Figures 1a–1f show the variation in the PDF over the cross section of a turbulent shear flow. The dominant features are two narrow peaks that correspond to the contributions to the scalar from the central (high-concentration) and peripheral (low-concentration) regions of the flow. The PDFs shown on Fig. 1 are consistent with the theoretical description of self-similar scalar PDFs within a turbulent shear flow that is developed in Chatwin and Sullivan[3] and which predicates on a separation of the time scales of long-term molecular diffusion and short-term local turbulent convective motions. A low-concentration peak progresses from a small magnitude at the center to virtually the entire PDF at the periphery of flows, whereas a high, dominant peak at the central region diminishes to insignificance in the peripheral regions. It is interesting to note in passing the proposal of Effelsberg and Peters,[4] and its discussion in Drake et al.,[5] to represent the PDF as a composite of a delta function (non-turbulent fluid), a Beta distribution (fully turbulent part), and an algebraic part for a transitional superlayer. The forms of the Beta distribution are Gaussian-like throughout. The disadvantages of that approach are discussed in Chatwin and Sullivan.[2]

It is of interest to compare the results shown in Fig. 1 with experimentally observed values presented in Antonia and Sreenivasan[6] (see also Dahm and Dimotakis[7]). There in a co-flowing, heated, round jet, scalar PDFs appeared to consist of a rather sharp spike near the nominal freestream temperature and a second much wider hump centered on higher values of temperature. The narrow low-temperature spike is of relatively small magnitude on the centerline and increases in magnitude, relative to the high-temperature mound, until it becomes the dominant feature of the PDF at the jet periphery. The two peaks are of equal magnitude at the approximate location of the mean temperature half-width. These observations of the low-temperature spike are certainly qualitatively similar to the distributions presented in Fig. 1 and the equal height of the two peaks at approximately the half-width location is observed in Fig. 1b. There is a significant difference between the observed high-temperature mound width and the corresponding narrow distribution shown on Fig. 1.

The inversion technique, whereby a finite number of moments given in Eq. (1) are used to compile the PDFs shown on Fig. 1, is described in Derksen and Sullivan.[1] Newton's

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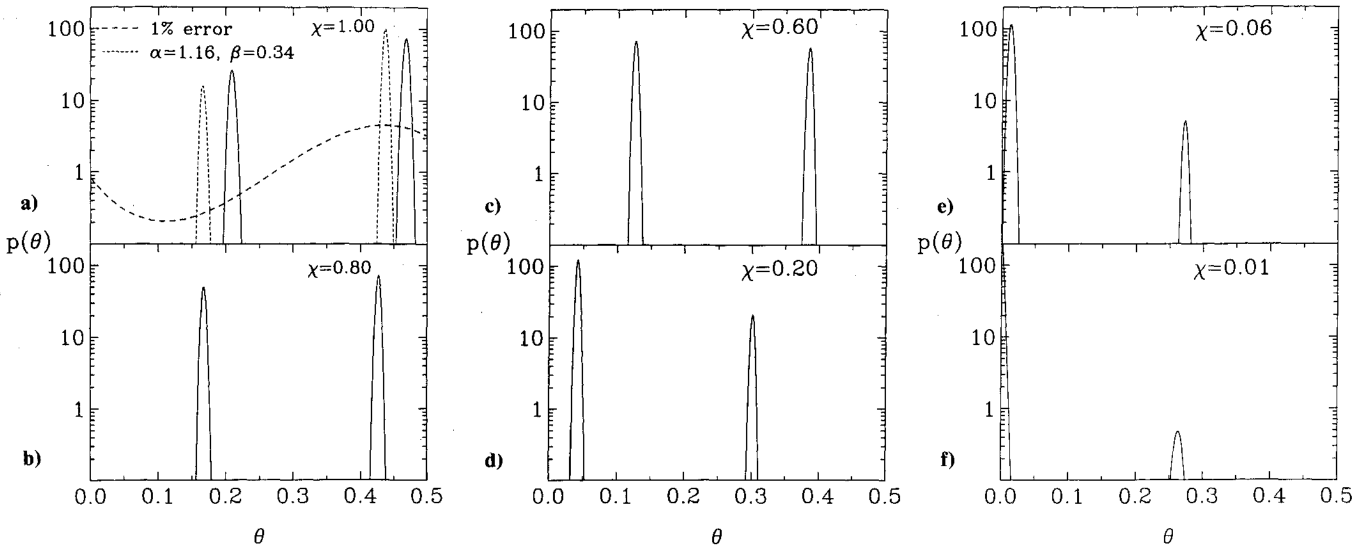

Fig. 1 Asymptotic scalar PDFs where θ is the concentration normalized with $2.5\mu_1(0)$. Except where otherwise indicated, all curves correspond to $\alpha = 1.36$ and $\beta = 0.23$, with the moments fit to within 1% of those given by Eq. (1).

method is used to solve for the coefficients of the polynomial $p_n(\theta)$ where

$$p(\theta) = \exp[p_n(\theta)] \tag{4}$$

and μ_n as defined in Eq. (2) are supplied from Eq. (1). Difficulties arise for distributions with narrow peaks from both establishing an accurate numerical integration scheme and from the solution of a system of equations that are not well conditioned. The singular value decomposition, as described in Press et al.,[8] was used to deal with the ill-conditioned matrices, and Simpson's rule with 1000 subintervals was found to give an adequate resolution of the integrals defining the moments. In this scheme the lowest four integral moments from Eq. (1) are prescribed, and as a necessary part of the process, the next four higher ordered moments are compiled. These higher moments invariably agree with the corresponding higher moments given directly by Eq. (1) to six decimal places, which is comparable with the numerical accuracy of the calculation. That is, these PDFs appear to be well described by a four parameter family. An investigation of higher moments in turbulent shear flows for scalar concentration, velocity, and wall shear stress appear to support this observation; however, measurement accuracy, particularly for odd moments, remains questionable. Thus the PDFs shown in Fig. 1 appear to be a proper representation of the moment set as given in Eq. (1) with α and β taken to be constant for all of the moments.

It is interesting to note the dashed curve shown in Fig. 1a. There the computation was terminated when only 1% convergence was established for the first four moments. The dashed PDF on the figure is a shape commonly observed (see, for example, Birch et al.[9]), and yet it is entirely an artifact of the procedure. The distinction between different self-similar turbulent shear flows and scalar injection configuration is encoded in the parameters α and β that appear in Eq. (1). A change in these parameters, within the normally small range of experimental values cited, does not appear to substantially alter the overall structure shown on Fig. 1.

Concluding Remarks

The distribution of moments as prescribed by Eq. (1) appeared in Chatwin and Sullivan to be a good approximation to the available experimental measurements, which, it should be noted, were not made with the specific validation of Eq. (1) as their objective. The PDFs that result from inversion of the moments of Eq. (1), then, represent a first approximation to the asymptotic PDF. It is expected that the approximation is improved by allowing for small perturbations, which depend on the moments, to the constant values of α and β in Eq. (1). One notes the significant difference in shape of the PDF given in Fig. 1a for moments that differ by only $\pm 1\%$ from those prescribed by Eq. (1) and the very good agreement between the inversion of measured moments with the measured PDFs discussed in Derksen and Sullivan.[1]

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References

[1]Derksen, R. W., and Sullivan, P. J., "Moment Approximations for Probability Density Functions," *Combustion and Flame,* Vol. 81, Sept. 1990, pp. 378–391.

[2]Chatwin, P. C., and Sullivan, P. J., "The Intermittency Factor of Scalars in Turbulence," *Physics of Fluids A,* Vol. 1, No. 4, 1989, pp. 761–763.

[3]Chatwin, P. C., and Sullivan, P. J., "A Simple and Unifying Physical Interpretation of Scalar Fluctuation Measurements from Many Turbulent Shear Flows," *Journal of Fluid Mechanics,* Vol. 212, March 1990, pp. 533–556.

[4]Effelsberg, E., and Peters, N., "A Composite Model for the Conserved Scalar PDF," *Combustion and Flame,* Vol. 50, April 1983, pp. 351–360.

[5]Drake, M. C., Shyy, W., and Pitz, R. W., "Superlayer Contributions to Conserved Scalar PDFs in an H_2 Turbulent Jet Diffusion Flame," *Proceedings of the 5th Symposium on Turbulent Shear Flows,* Cornell Univ., Ithaca, NY, 1985, pp. 10.13–10.18.

[6]Antonia, R. A., and Sreenivasan, K. R., "Statistical Properties of Velocity and Temperature Fluctuations in a Turbulent Heated Jet," Dept. of Mechanical Engineering, Univ. of Newcastle, TNFM 3, New South Wales, Australia, 1976.

[7]Dahm, D. J. A., and Dimotakis, P. E., "Mixing at Large Schmidt Number in the Self Similar Far Field of Turbulent Jets," *Journal of Fluid Mechanics,* Vol. 217, Aug. 1990, pp. 299–330.

[8]Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., *Numerical Recipes: The Art of Scientific Computing,* Cambridge Univ. Press, Cambridge, England, UK, 1986, pp. 52–64.

[9]Birch, A. D., Brown, A. D., Dodson, D. R., and Thomas, J. R., "The Turbulent Concentration Field of a Methane Jet," *Journal of Fluid Mechanics,* Vol. 88, Pt. 3, 1978, pp. 431–449.